A THEOREM IN COMBINATIONS.

By J. V. Westfall, Ph. D.
Instructor in Mathematics.

In determining the probability of the happening of a certain event in the game of Nim, the theory of which was published by C. L. Bouton in the *Annals of Mathematics*, I had occasion to determine the number of ways in which a pile of counters all alike may be divided into three piles. In case the number is small the problem can be solved by trial with little difficulty but even with a number of moderate size, e. g. 30, this method becomes somewhat tedious. The following discussion leads to two general formulæ which will give the desired result simply and quickly, however large the number may be.

The problem is divided into two cases according as the number of counters is odd or even. Suppose first the number to be even and equal to $2n$. The following scheme will then readily give us an expression for the number of ways in which the counters can be divided.

The Roman numerals at the heads of the columns indicate the number of the pile, and the numbers in the columns indicate the number in each pile corresponding to the different divisions.

$$\begin{array}{c} \begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{III} \end{array} \\ \left\{\begin{array}{ccc} 1 & 1 & 2n-2 \\ 1 & 2 & 2n-3 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 1 & n-1 & n \end{array}\right\} \end{array}$$

$$
\begin{array}{ccc}
\text{I} & \text{II} & \text{III}
\end{array}
$$

$$
\left\{\begin{array}{ccc}
2 & 2 & 2n-4 \\
2 & 3 & 2n-5 \\
\vdots & \vdots & \vdots \\
2 & n-1 & n-1
\end{array}\right\}
$$

$$
\left\{\begin{array}{ccc}
3 & 3 & 2n-6 \\
3 & 4 & 2n-7 \\
\vdots & \vdots & \vdots \\
3 & n-2 & n-1
\end{array}\right\}
$$

$$
\left\{\begin{array}{ccc}
2r-1 & 2r-1 & 2n-4r+2 \\
2r-1 & 2r & 2n-4r+1 \\
\vdots & \vdots & \vdots \\
2r-1 & n-r & n-r+1
\end{array}\right\}
$$

$$
\left\{\begin{array}{ccc}
2r & 2r & 2n-4r \\
2r & 2r+1 & 2n-4r-1 \\
\vdots & \vdots & \vdots \\
2r & n-r & n-r
\end{array}\right\}
$$

$$
\left\{\begin{array}{ccc}
2r+1 & 2r+1 & 2n-4r-2 \\
2r+1 & 2r+2 & 2n-4r-3 \\
\vdots & \vdots & \vdots \\
2r+1 & n-r-1 & n-r
\end{array}\right\}
$$

It is easily seen that there are $n-1$ ways of dividing the counters in which one pile always contains one counter, $n-2$ ways in which one pile always contains two counters and no pile less than two, and $n-4$ ways in which one pile always contains three counters and no pile less than three. In general there are $n - r + 2$ ways in which one pile always contains $2r - 1$ counters and no pile less than $2r - 1$, $n - 3r + 1$ ways in which one pile always contains $2r$ counters and no pile less than $2r$, and $n - 3r - 1$ ways in which one pile always contains $2r + 1$ counters and no pile less than $2r + 1$. As the number in the first pile is successively increased by one the number of new ways decreases alternately by one and two. We will finally arrive at the point where the number of new ways will be either one or two depending upon n. It is evident then that this will exhaust the different possible divisions.

Denote the number of possible divisions by S_{2n}. S_{2n} is then the sum of the positive terms of the series

$$(n - 1) + (n - 2) + (n - 4) + (n - 5) + \ldots\ldots$$

i. e. $S_{2n} = A_1 + A_2$, where A_1 and A_2 are respectively the sums of the positive terms of the two arithmetical progressions

$$(n - 1) + (n - 4) + (n - 7) + \ldots\ldots \text{ and}$$
$$(n - 2) + (n - 5) + (n - 8) + \ldots\ldots$$

It follows then that

$$A_1 = (n - 1) + (n - 4) + (n - 7) + \ldots\ldots \text{ to } x \text{ terms}$$

where x is the largest positive integer which satisfies the condition that $n - 3x + 2$ shall be greater than zero, and

$$A_2 = (n - 2) + (n - 5) + (n - 8) + \ldots\ldots \text{ to } x \text{ terms}$$

where x has the same value as in A_1 . $n - 3x + 1$ being not less than zero when $n - 3x + 2$ is greater than zero.

Therefore

$$S_{2n} = \frac{x}{2}(2n - 3x + 1) + \frac{x}{2}(2n - 3x - 1)$$
$$= x(2n - 3x)$$

$$\begin{array}{ccc} \text{I} & \text{II} & \text{III} \\ \end{array}$$
$$\left\{\begin{array}{ccc} 1 & 1 & 2n - 1 \\ 1 & 2 & 2n - 2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & n & n \end{array}\right]$$

$$\begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{III} \end{array}$$

$$\left\{\begin{array}{ccc} 2 & 2 & 2n-3 \\ 2 & 3 & 2n-4 \\ \vdots & \vdots & \vdots \\ 2 & n-1 & n \end{array}\right\}$$

$$\left\{\begin{array}{ccc} 3 & 3 & 2n-5 \\ 3 & 4 & 2n-6 \\ \vdots & \vdots & \vdots \\ 3 & n-1 & n-1 \end{array}\right\}$$

$$\left\{\begin{array}{ccc} 2r-1 & 2r-1 & 2n-4r+3 \\ 2r-1 & 2r & 2n-4r+2 \\ \vdots & \vdots & \vdots \\ 2r-1 & n-r+1 & n-r+1 \end{array}\right\}$$

$$\left\{\begin{array}{ccc} 2r & 2r & 2n-4r+1 \\ 2r & 2r+1 & 2n-4r \\ \vdots & \vdots & \vdots \\ 2r+1 & n-r & n-r+1 \end{array}\right\}$$

$$\left\{\begin{array}{ccc} 2r+1 & 2r+1 & 2n-4r-1 \\ 2r+1 & 2r+2 & 2n-4r-2 \\ \vdots & \vdots & \vdots \\ 2r+1 & n-r & n-r \end{array}\right\}$$

It is easily seen as before that

$$S_{2n+1} = A_2 + A_3$$

where A_2 is the same series as before, and

$$A_3 = n + (n-3) + (n-6) + \ldots\ldots \text{ to } y \text{ terms,}$$

where y is the largest positive integer which satisfies the condition that $n - 3y + 3$ shall be greater than zero.

It follows then that

$$S_{2n+1} = \frac{y}{2}(2n - 3y + 3) + \frac{x}{2}(2n - 3x - 1)$$

Recapitulating we have:

$S_{2n} = x(2n - 3x)$, where x is the largest positive interger which satisfies the condition $n - 3x + 2$ shall be greater than zero, and

$$S_{2n+1} = \frac{y}{2}(2n - 3y + 3) + \frac{x}{2}(2n - 3x - 1)$$

where y is the largest positive integer which satisfies the condition that $n - 3y + 3$ shall be greater than zero, and x is determined as before.